Linear conductance in Coulomb-blockade quantum dots in the presence of interactions and spin

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We discuss the calculation of the linear conductance through a Coulomb-blockade quantum dot in the presence of interactions beyond the charging energy. In the limit where the temperature is large compared with a typical tunneling width, we use a rate-equations approach to describe the transitions between the corresponding many-body states. We discuss both the elastic and rapid-thermalization limits, where the rate of inelastic scattering in the dot is either small or large compared with the elastic transition rate, respectively. In the elastic limit, we find several cases where a closed solution for the conductance is possible, including the case of a constant exchange interaction. In the rapid-thermalization limit, a closed solution is possible in the general case. We show that the corresponding expressions for the linear conductance simplify for a Hamiltonian that is invariant under spin rotations.

I. INTRODUCTION

In the Coulomb-blockade regime of quantum dots, the conductance of the dot exhibits peaks as a function of the gate voltage, where each peak corresponds to the tunneling of one more electron into the dot. Of particular interest are diffusive or ballistic chaotic dots, where the mesoscopic fluctuations of the conductance peaks and their spacings reflect the statistical nature of the ground-state wave function and energy of the isolated dot.[1,2] The simplest model to describe Coulomb blockade is the constant-interaction (CI) model, in which the electrons occupy single-particle levels in the dot and the interaction is described by an electrostatic Coulomb energy that is constant for a fixed number of electrons. In this model, the conductance near a Coulomb-blockade peak at temperatures that are large compared with a typical tunneling width can be derived in a closed form using a master-equations approach.[3,4] This regime of sequential tunneling is relevant to most Coulomb-blockade experiments involving weakly-coupled dots.

At sufficiently low temperatures, the conductance through the dot is dominated by elastic processes. However, at finite temperature, the electrons in the dot may undergo inelastic scattering processes caused, for example, by electron-phonon or electron-electron interactions. In general, one may consider two limiting cases, depending on the relative magnitude of a typical elastic tunneling width $\Gamma_{\rm el}$ of an electron into the dot, and a typical inelastic scattering width $\Gamma_{\rm in}$ of electrons in the dot. In the so-called elastic limit, $\Gamma_{\rm el} \gg \Gamma_{\rm in}$, and inelastic scattering processes in the dot can be ignored. In the opposite limit, $\Gamma_{\rm el} \ll \Gamma_{\rm in}$, inelastic scattering occurs on such a short time scale that electrons in the dot are effectively thermalized immediately after an electron tunnels in or out of the dot.[4] This limit will be called the rapid-thermalization

limit, which in the literature is also sometimes referred to as the inelastic limit. Statistical properties of the conductance peaks were studied within the CI model in both elastic[5,6] and rapid-thermalization[7,8,9] limits, as well as for intermediate situations.[10]

However, recent experiments in almost-isolated Coulomb-blockade dots clearly indicate that it is necessary to take into account electron-electron interactions beyond the CI model.[11] For such interactions, the eigenstates of the dot are no longer characterized by Slater determinants. An important question that arises is how to calculate the linear conductance through such a dot. In this work we provide an answer in the limit of sequential tunneling. We make the general assumption that both the thermal energy and the typical excitation energies in the dot are small compared with the charging energy. This defines the regime of Coulomb blockade, where in the vicinity of the $N + 1$-st conductance peak only the manifolds of states with N and $N + 1$ electrons in the dot contribute to the conductance (the manifolds with an electron number different from either N and $N + 1$ are pushed away by the charging energy). Given the many-body eigenstates of the dot (with N and $N + 1$ electrons), their energies, and the transition widths between the states in the N- and $(N + 1)$-electron dot as an electron tunnels into the dot, we discuss the calculation of the linear conductance. This is done by introducing non-equilibrium probabilities of the dot to be in its various many-body states. These probabilities satisfy a set of rate equations describing the transitions between the many-body eigenstates. We consider stationary solutions in the linear-response approximation in both limits where inelastic scattering of electrons in the dot is weak (elastic limit) and strong (rapid-thermalization limit). In the elastic limit, the linearized rate equations cannot be solved in closed form in the general case, but there are

several important cases where a closed solution exits. Such cases include sufficiently low temperatures where only ground-state to ground-state, or ground-state to excited-states transitions (of N and $N + 1$ electrons, respectively) contribute to the conductance. Another case where the rate equations can be solved in closed form at any temperature corresponds to a Hamiltonian for which the occupations of the orbital single-particle states are good quantum numbers. A particularly important such case corresponds to the universal Hamiltonian[2,12] in the limit of infinite Thouless conductance. This Hamiltonian includes a constant exchange interaction in addition to a constant charging energy.[13] We also consider the rapid-thermalization limit, where we can always obtain a closed form expression for the conductance.

Furthermore, when the Hamiltonian of the dot is spin-rotation invariant (as is the case of the Coulomb interaction), the expressions we find for the linear conductance can be simplified by carrying out explicitly the sum over the magnetic quantum numbers (in both the elastic and rapid-thermalization limits).

The outline of this paper is as follows: In Section II we introduce the basic assumptions and notation. In Section III we discuss the rate equations satisfied by the probabilities to find the dot in one of its many-body eigenstates with N or $N + 1$ electrons. We assume the elastic limit, where inelastic scattering of electrons in the dot is negligible. In Section IV we use linear-response theory to linearize these master equations in the stationary limit. A general expression for the linear conductance in terms of the solution to the rate equations is derived in Section V. We show that if detailed balance is satisfied for each pair of many-body eigenstates [in the N- and $(N + 1)$-electron dots, respectively], then the master equations and consequently the conductance can be solved in closed form. Furthermore, the corresponding expressions are shown to be simplified for Hamiltonians that are invariant with respect to spin rotations. Physically relevant cases where such closed solutions exist are discussed in Section VI, and include the case of the universal Hamiltonian in the limit of infinite Thouless conductance. In Section VII we solve the rate equations and derive a closed form for the conductance in the case where the inelastic scattering width is much larger than the elastic width (rapid-thermalization limit). Finally, in Section VIII we summarize the main results of this paper.

II. MODEL

We consider an almost-isolated quantum dot described by a Hamiltonian $\hat{H}$ that includes a one-body part and a two-body interaction. The dot is weakly coupled to leads, and we assume the limit $kT \gg \Gamma$, where Γ is a typical transition width of an electron from the leads into the dot. In this limit we can ignore the coherence between the dot and the leads and use a rate-equations approach

to study the transport through the dot. Furthermore, we assume the Coulomb energy e^2/C to be much larger than the thermal energy kT. In this limit, the linear conductance of the dot exhibits Coulomb-blockade peaks. When the gate voltage V_g is tuned in the vicinity of a Coulomb-blockade peak, the number of electrons in the dot can be either N or $N + 1$. All manifolds with an electron number different from either N or $N + 1$ are pushed away by the charging energy.

The rate equations have been solved in the CI model for a single spin-degenerate level[3] and for any number of single-particle levels.[4] In the CI model, the two-body interaction is modeled by a constant charging energy of an object with capacitance C and charge Ne. Here, we derive and solve the rate equations in the presence of interactions beyond the charging energy, where the many-body states of the dots are no longer Slater determinants.

We denote the many-body eigenstates of the dot with N electrons by i, and their respective energies by $\varepsilon_i^{(N)}$. Similarly, the states of the dot with $N + 1$ electrons are labeled as j and the corresponding energies are $\varepsilon_j^{(N+1)}$. M_N and M_{N+1} denote, respectively, the total number of the N- and $(N + 1)$-electron states considered.

Of particular interest are cases where the dot's Hamiltonian is spin-rotation invariant. This requires the absence of a spin-orbit interaction such that the single-particle energies of the electrons in the dot ϵ_λ are spin-degenerate. In addition, for spin-rotation invariance to be valid, the matrix elements of the two-body interaction may depend only on the orbital states but not on the spin indices, as it is the case for the Coulomb interaction. We then have

$$[\hat{H}, \hat{\mathbf{S}}^2] = 0 \quad \text{and} \quad [\hat{H}, \hat{S}_z] = 0 . \tag{1}$$

The many-body eigenstates of the dot can be characterized by the good quantum numbers of the spin S and its projection $S_z = M$. The remaining quantum numbers, in addition to the total number of electrons N, will be labeled by α. Thus the eigenstates of the dot with n electrons are given by $i = (\alpha, S, M)$, and their respective energies $\varepsilon_{\alpha S}^{(N)}$ are independent of M. The states of the dot with $N + 1$ electrons are similarly labeled by $j = (\alpha', S', M')$, and their corresponding energies are $\varepsilon_{\alpha'S'}^{(N+1)}$. An example of a spin-rotation invariant Hamiltonian is the recently proposed universal Hamiltonian.[2,12]

III. RATE EQUATIONS

We assume that a potential difference V is applied between the two leads (source and drain) at temperature T. Fractions η_l and η_r of this potential difference, with

$$\eta_l + \eta_r = 1 , \tag{2}$$

fall between the dot and the left lead, and the right lead and the dot, respectively. As a result a current I flows

through the dot and various eigenstates of the dot can be occupied with different probabilities. We denote the non-equilibrium probability for the dot to be in a particular n-electron state i by $P_i^{(N)}$. Since we consider situations in which the dot can only be occupied by N or $N + 1$ electrons, we require the normalization

$$\sum_i P_i^{(N)} + \sum_j P_j^{(N+1)} = 1 \,. \qquad (3)$$

We further assume that energy is conserved in the tunneling between the dot and the leads, neglecting vir-

tual transitions that are of higher order in the tunneling widths. Denoting the transition widths between the N-electron state i and the $(N + 1)$-electron state j involving an electron tunneling into the dot from the left (right) lead by Γ_{ij}^{l} (Γ_{ij}^{r}), we can write the following rate equations for the probabilities $P_i^{(N)}$ and $P_j^{(N+1)}$:

$$\frac{\partial P_i^{(N)}}{\partial t} = \sum_j P_j^{(N+1)}[(1 - f_{ij}^{\mathrm{l}})\Gamma_{ij}^{\mathrm{l}} + (1 - f_{ij}^{\mathrm{r}})\Gamma_{ij}^{\mathrm{r}}] - P_i^{(N)} \sum_j (f_{ij}^{\mathrm{l}}\Gamma_{ij}^{\mathrm{l}} + f_{ij}^{\mathrm{r}}\Gamma_{ij}^{\mathrm{r}}) \quad \text{for all } i \,, \qquad (4\mathrm{a})$$

$$\frac{\partial P_j^{(N+1)}}{\partial t} = \sum_i P_i^{(N)}(f_{ij}^{\mathrm{l}}\Gamma_{ij}^{\mathrm{l}} + f_{ij}^{\mathrm{r}}\Gamma_{ij}^{\mathrm{r}}) - P_j^{(N+1)} \sum_j [(1 - f_{ij}^{\mathrm{l}})\Gamma_{ij}^{\mathrm{l}} + (1 - f_{ij}^{\mathrm{r}})\Gamma_{ij}^{\mathrm{r}}] \quad \text{for all } j \,. \qquad (4\mathrm{b})$$

The Fermi-Dirac function of an electron in the left (right) lead is evaluated at a suitable energy that can be found from energy conservation of the transition between states i and j, and is denoted by f_{ij}^{l} (f_{ij}^{r}). In the presence of a gate voltage V_{g}, the energy of the n-electron dot in state i is given by $\varepsilon_i^{(N)} - Ne\zeta V_{\mathrm{g}}$. Here $\zeta \equiv C_{\mathrm{g}}/C$, with C_{g} being the dot capacitance with respect to the gate, and C being the full dot capacitance. Thus the energy of an electron (relative to the Fermi energy ϵ_{F} in the leads) in the left (right) lead that tunnels into an N-electron dot in state i to form an $(N + 1)$-electron dot in state j is given by

$$\varepsilon_{ij} = \varepsilon_j^{(N+1)} - \varepsilon_i^{(N)} - \tilde{\epsilon}_{\mathrm{F}} \,, \qquad (5)$$

where $\tilde{\epsilon}_{\mathrm{F}} \equiv e\zeta V_{\mathrm{g}} + \epsilon_{\mathrm{F}}$ is an effective Fermi energy. Taking into account the bias-potential drop between the dot and each lead, we have

$$f_{ij}^{\mathrm{l}} \equiv f(\varepsilon_{ij} + \eta_{\mathrm{l}} eV) \quad \text{and} \quad f_{ij}^{\mathrm{r}} \equiv f(\varepsilon_{ij} - \eta_{\mathrm{r}} eV) \,, \qquad (6)$$

where the Fermi-Dirac function is defined by $f(x) = [1 + \exp(\beta x)]^{-1}$ with $\beta = 1/kT$.

We note that we can choose any values of η_{l} and η_{r} satisfying Eq. (2), but that the final result for the physical quantities (e.g. the conductance) must be independent of this choice (see the Appendix for a proof).

IV. LINEAR RESPONSE

To calculate the conductance, we are interested in finding stationary solutions of Eqs. (4), i.e. non-equilibrium probabilities $P_i^{(N)}$ and $P_j^{(N+1)}$ that are independent in

time,

$$\frac{\partial P_i^{(N)}}{\partial t} = 0 \ \text{ for all } i \,, \qquad (7\mathrm{a})$$

$$\frac{\partial P_j^{(N+1)}}{\partial t} = 0 \ \text{ for all } j \,. \qquad (7\mathrm{b})$$

Furthermore, to calculate the linear conductance, we are interested in calculating the current for small bias voltages V, where the equations can be solved in linear-response theory. To that end, we linearize Eqs. (4) in V. This involves an expansion of the Fermi-Dirac distribution functions f_{ij}^{l}, f_{ij}^{r} as well as the probabilities $P_i^{(N)}$ and $P_j^{(N+1)}$ around their equilibrium values. The Fermi-Dirac functions are expanded up to first order as

$$f_{ij}^{\mathrm{l}} = f_{ij} + \eta_{\mathrm{l}} eV f_{ij}' \quad \text{and} \quad f_{ij}^{\mathrm{r}} = f_{ij} - \eta_{\mathrm{r}} eV f_{ij}' \,, \qquad (8)$$

where we used the short-hand notation $f_{ij} = f(\varepsilon_{ij})$. The probabilities $P_i^{(N)}$ and $P_j^{(N+1)}$ are expanded around their equilibrium values $\tilde{P}_i^{(N)}$ and $\tilde{P}_j^{(N+1)}$ which they would attain if the bias voltage would be zero and the dot would be in thermal equilibrium,

$$P_i^{(N)} = \tilde{P}_i^{(N)}[1 + eV\beta\Psi_i^{(N)}] \,, \qquad (9\mathrm{a})$$

$$P_j^{(N+1)} = \tilde{P}_j^{(N+1)}[1 + eV\beta\Psi_j^{(N+1)}] \,, \qquad (9\mathrm{b})$$

in terms of new variables $\Psi_i^{(N)}$ and $\Psi_j^{(N+1)}$. The equilibrium probabilities can be expressed explicitly in terms of the eigenenergies $\varepsilon_i^{(N)}$ and $\varepsilon_j^{(N+1)}$ of the dot with N and $N + 1$ electrons,

$$\tilde{P}_i^{(N)} = \frac{e^{-\beta(\varepsilon_i^{(N)} - \tilde{\epsilon}_{\mathrm{F}} N)}}{Z} \qquad (10\mathrm{a})$$

and

$$\tilde{P}_j^{(N+1)} = \frac{e^{-\beta[\varepsilon_j^{(N+1)} - \tilde{\epsilon}_{\mathrm{F}}(N+1)]}}{Z} \,, \qquad (10b)$$

with the partition sum

$$Z = \sum_i e^{-\beta(\varepsilon_i^{(N)} - \tilde{\epsilon}_{\mathrm{F}} N)} + \sum_j e^{-\beta[\varepsilon_j^{(N+1)} - \tilde{\epsilon}_{\mathrm{F}}(N+1)]} \,. \quad (11)$$

The equilibrium distribution is in principle grand-canonical with a chemical potential equal to the Fermi energy ϵ_{F} (at temperature T) in the leads, but because of the presence of charging energy only two values of the electron numbers are allowed. To find a set of equations for the new variables $\Psi_i^{(N)}$ and $\Psi_j^{(N+1)}$, we substitute the expansions (8) and (9) into the r.h.s. of the rate equations (4) and equate the results to zero to ensure stationarity. Keeping only the terms linear in the bias voltage V, we obtain

$$\sum_j \tilde{P}_j^{(N+1)} [\beta \Psi_j^{(N+1)}(1 - f_{ij})(\Gamma_{ij}^{\mathrm{l}} + \Gamma_{ij}^{\mathrm{r}}) - f'_{ij}(\eta_{\mathrm{l}}\Gamma_{ij}^{\mathrm{l}} - \eta_{\mathrm{r}}\Gamma_{ij}^{\mathrm{r}})]$$

$$= \tilde{P}_i^{(N)} [\beta \Psi_i^{(N)} \sum_j f_{ij}(\Gamma_{ij}^{\mathrm{l}} + \Gamma_{ij}^{\mathrm{r}}) + \sum_j f'_{ij}(\eta_{\mathrm{l}}\Gamma_{ij}^{\mathrm{l}} - \eta_{\mathrm{r}}\Gamma_{ij}^{\mathrm{r}})] \quad \text{for all } i \,, \qquad (12a)$$

$$\sum_i \tilde{P}_i^{(N)} [\beta \Psi_i^{(N)} f_{ij}(\Gamma_{ij}^{\mathrm{l}} + \Gamma_{ij}^{\mathrm{r}}) - f'_{ij}(\eta_{\mathrm{l}}\Gamma_{ij}^{\mathrm{l}} - \eta_{\mathrm{r}}\Gamma_{ij}^{\mathrm{r}})]$$

$$= \tilde{P}_j^{(N+1)} [\beta \Psi_j^{(N+1)} \sum_i (1 - f_{ij})(\Gamma_{ij}^{\mathrm{l}} + \Gamma_{ij}^{\mathrm{r}}) - \sum_i f'_{ij}(\eta_{\mathrm{l}}\Gamma_{ij}^{\mathrm{l}} - \eta_{\mathrm{r}}\Gamma_{ij}^{\mathrm{r}})] \quad \text{for all } j \,. \qquad (12b)$$

The above equations can be simplified utilizing the following three relations. From the definition (10) of the equilibrium probabilities, it follows that

$$\tilde{P}_j^{(N+1)} = \tilde{P}_i^{(N)} e^{-\beta \varepsilon_{ij}} \,, \qquad (13)$$

while the Fermi-Dirac distribution satisfies

$$[1 - f(x)]e^{-\beta x} = f(x) \qquad (14)$$

and

$$f'(x) = -\beta f(x)/[1 + \exp(-\beta x)] \,. \qquad (15)$$

Using Eqs. (13), (14), and (15), we arrive at the transformed rate equations

$$\sum_j f_{ij}[(\Gamma_{ij}^{\mathrm{l}} + \Gamma_{ij}^{\mathrm{r}})(\Psi_j^{(N+1)} - \Psi_i^{(N)}) + (\eta_{\mathrm{l}}\Gamma_{ij}^{\mathrm{l}} - \eta_{\mathrm{r}}\Gamma_{ij}^{\mathrm{r}})] = 0 \quad \text{for all } i \,, \qquad (16a)$$

$$\sum_i (1 - f_{ij})[(\Gamma_{ij}^{\mathrm{l}} + \Gamma_{ij}^{\mathrm{r}})(\Psi_j^{(N+1)} - \Psi_i^{(N)}) + (\eta_{\mathrm{l}}\Gamma_{ij}^{\mathrm{l}} - \eta_{\mathrm{r}}\Gamma_{ij}^{\mathrm{r}})] = 0 \quad \text{for all } j \,. \qquad (16b)$$

Equations (16) represent a system of $M_N + M_{N+1}$ linear equations for $M_N + M_{N+1}$ unknowns. However, only $M_N + M_{N+1} - 1$ of these equations are linearly independent. Hence, the solutions for $\Psi_i^{(N)}$ and $\Psi_j^{(N+1)}$ are determined only up to an additive constant. This constant is fixed by imposing the constraint

$$\sum_i \tilde{P}_i^{(N)} \Psi_i^{(N)} + \sum_j \tilde{P}_j^{(N+1)} \Psi_j^{(N+1)} = 0 \,, \qquad (17)$$

that follows from the normalization condition (3). In general, it is necessary to solve Eqs. (16) numerically. These equations represent sets of detailed-balance equations. However, it is possible to find analytic solutions in cases where a detailed-balance equation holds for each pair of states i and j individually, i.e. in cases where each term in the square brackets of Eqs. (16) vanishes. This

condition is equivalent to

$$\Psi_i^{(N)} - \Psi_j^{(N+1)} = \frac{\eta_l \Gamma_{ij}^l - \eta_r \Gamma_{ij}^r}{\Gamma_{ij}^l + \Gamma_{ij}^r} \quad \text{for any } i \text{ and } j . \quad (18)$$

We emphasize that there are $M_N M_{N+1}$ equations (18) for only $M_N + M_{N+1}$ unknowns, so in the general case it is not possible to satisfy (18). However, there are a number of important cases where a solution does exist, as we discuss in Section VI. A solution to Eqs. (18), when it exists, does not depend on temperature and leads to an expression for the conductance in which the dependence on temperature and on decay widths factorizes.

V. LINEAR CONDUCTANCE

The linear conductance G is defined as the ratio I/V in the limit $V \rightarrow 0$. The current through the dot can be expressed in terms of the quantities considered in the master-equation approach. In this framework, we can calculate for example the current I through the left tunneling contact which equals the current through the dot and through the right contact,

$$I = \frac{e}{\hbar} \sum_{ij} [P_i^{(N)} f_{ij}^l - P_j^{(N+1)} (1 - f_{ij}^l)] \Gamma_{ij}^l . \quad (19)$$

Substituting the expansions in Eqs. (8) and (9) into Eq. (19), we find the elastic linear conductance

$$G_{el} = \frac{e^2}{\hbar k T} \sum_{ij} \tilde{P}_i^{(N)} f_{ij} (\Psi_i^{(N)} - \Psi_j^{(N+1)} - \eta_l) \Gamma_{ij}^l . \quad (20)$$

Although η_l appears explicitly in Eq. (20), the linear conductance is independent of η_l. This must be so on physical grounds, and a formal proof is provided in the Appendix.

In general, we have to solve Eqs. (16) for $\Psi_i^{(N)}$ and $\Psi_j^{(N+1)}$ and substitute their values in Eq. (20). However, in the special case where Eqs. (18) hold, the conductance

is given by the closed form

$$G_{el} = \frac{e^2}{\hbar k T} \sum_{ij} \tilde{P}_i^{(N)} f_{ij} \frac{\Gamma_{ij}^l \Gamma_{ij}^r}{\Gamma_{ij}^l + \Gamma_{ij}^r} , \quad (21a)$$

or equivalently

$$G_{el} = \frac{e^2}{\hbar k T} \sum_{ij} \tilde{P}_j^{(N+1)} (1 - f_{ij}) \frac{\Gamma_{ij}^l \Gamma_{ij}^r}{\Gamma_{ij}^l + \Gamma_{ij}^r} . \quad (21b)$$

Notice that the parameters $\eta_{l,r}$ no longer appear in the conductance formula. The independence of the conductance from the values of $\eta_{l,r}$, which holds true in general (cf. Appendix), becomes apparent in this special case where an explicit formula can be given. For the conductance in Eqs. (21), the temperature dependence of the contribution from each pair of states in the N-electron and $(N + 1)$-electron dots is independent from the transition widths.

So far, the spin symmetries have not been taken into account in the derivation of the rate equations and the conductance. If this is done, the sum over the many-body states i and j can be further simplified. For two many-body states $i = (\alpha, S, M)$ and $j = (\alpha', S', M')$, the associated partial decay widths Γ_{ij}^l and Γ_{ij}^r are given by

$$\Gamma_{ij}^{l,r} = \Gamma_0^{l,r} |\langle \alpha' S' M' | \psi_m^\dagger (\mathbf{r}_{l,r}) | \alpha S M \rangle|^2 , \quad (22)$$

where the operator $\psi_m^\dagger(\mathbf{r}_l)$ $[\psi_m^\dagger(\mathbf{r}_r)]$ creates an electron with spin projection m at the left (right) point contact at $\mathbf{r}_l$ ($\mathbf{r}_r$), and we have introduced overall coupling strengths $\Gamma_0^{l,r}$ at the left and right point contact. The spin selection rules require $m = M' - M$ and $S' = |S \pm 1/2|$ for the matrix element in Eq. (22) not to vanish. For a Hamiltonian which is invariant with respect to spin rotations, we can use the Wigner-Eckart theorem to factorize the matrix element in Eq. (22) into a Clebsch-Gordan (CG) coefficient and a reduced matrix element that is independent of the spin-projection quantum numbers M, M', and m,

$$\begin{aligned}
\langle \alpha' S' M' | \psi_m^\dagger(\mathbf{r}) | \alpha S M \rangle &= -\frac{1}{\sqrt{2S'+1}} (S\, M\, 1/2\, m | S'\, M') (\alpha' S' \| \psi^\dagger(\mathbf{r}) \| \alpha S) \\
&= (-1)^{S'-M'} \begin{pmatrix} S' & 1/2 & S \\ M' & m & M \end{pmatrix} (\alpha' S' \| \psi^\dagger(\mathbf{r}) \| \alpha S) .
\end{aligned} \quad (23)$$

For completeness, we have given the relation in Eq. (23) both in terms of a CG coefficient (first line) and a Wigner $3j$-symbol (second line). For a given pair of many-body states i and j, the decay widths Γ_{ij}^l and Γ_{ij}^r involve the same CG coefficient. In Eq. (18), the CG coefficients in the ratio on the r.h.s. cancel and we are left with

$$\Psi_i^{(N)} - \Psi_j^{(N+1)} = \frac{\eta_l \Gamma_0^l |(\alpha' S' \| \psi^\dagger(\mathbf{r}_l) \| \alpha S)|^2 - \eta_r \Gamma_0^r |(\alpha' S' \| \psi^\dagger(\mathbf{r}_r) \| \alpha S)|^2}{\Gamma_0^l |(\alpha' S' \| \psi^\dagger(\mathbf{r}_l) \| \alpha S)|^2 + \Gamma_0^r |(\alpha' S' \| \psi^\dagger(\mathbf{r}_r) \| \alpha S)|^2} . \quad (24)$$

Thus, if solutions for $\Psi_i^{(N)}$ and $\Psi_j^{(N+1)}$ exist that satisfy Eq. (18), they will be independent of the spin projec-

tion M or M', i.e. $\Psi_i^{(N)} = \Psi_{\alpha S}^{(N)}$ and $\Psi_j^{(N+1)} = \Psi_{\alpha'S'}^{(N+1)}$. Defining reduced decay widths

$$\tilde{\Gamma}_{\alpha S,\alpha'S'}^{\mathrm{l,r}} = \Gamma_0^{\mathrm{l,r}}|(\alpha'S'\|\psi^\dagger(\mathbf{r}_{\mathrm{l,r}})\|\alpha S)|^2 , \qquad (25)$$

conditions (24) can be rewritten in a form in which the magnetic quantum numbers disappear,

$$\Psi_{\alpha'S'}^{(N+1)} - \Psi_{\alpha S}^{(N)} = \frac{\eta_{\mathrm{l}}\tilde{\Gamma}_{\alpha S,\alpha'S'}^{\mathrm{l}} - \eta_{\mathrm{r}}\tilde{\Gamma}_{S,\alpha'S'}^{\mathrm{r}}}{\tilde{\Gamma}_{\alpha S,\alpha'S'}^{\mathrm{l}} + \tilde{\Gamma}_{\alpha S,\alpha'S'}^{\mathrm{r}}} . \qquad (26)$$

Using the reduced widths in the expression (21) for the linear conductance, we have

$$G_{\mathrm{el}} = \frac{e^2}{\hbar kT} \sum_{\alpha S\alpha'S'} \sum_{MM'm} \tilde{P}_{\alpha S}^{(N)} f(\varepsilon_{\alpha S,\alpha'S'}) \frac{1}{2S'+1}|(S\,M\,1/2\,m|S'\,M')|^2 \frac{\tilde{\Gamma}_{\alpha S,\alpha'S'}^{\mathrm{l}}\tilde{\Gamma}_{\alpha S,\alpha'S'}^{\mathrm{r}}}{\tilde{\Gamma}_{\alpha S,\alpha'S'}^{\mathrm{l}} + \tilde{\Gamma}_{\alpha S,\alpha'S'}^{\mathrm{r}}} . \qquad (27)$$

Taking advantage of the unitarity of the CG, $\sum_{Mm}|(S\,M\,1/2\,m|S'\,M')|^2 = 1$, we can write as our final result for the linear conductance

$$G_{\mathrm{el}} = \frac{e^2}{\hbar kT} \sum_{\alpha S\alpha'S'} \tilde{P}_{\alpha S}^{(N)} f(\varepsilon_{\alpha'S',\alpha S}) \frac{\tilde{\Gamma}_{\alpha S,\alpha'S'}^{\mathrm{l}}\tilde{\Gamma}_{\alpha S,\alpha'S'}^{\mathrm{r}}}{\tilde{\Gamma}_{\alpha S,\alpha'S'}^{\mathrm{l}} + \tilde{\Gamma}_{\alpha S,\alpha'S'}^{\mathrm{r}}} . \qquad (28)$$

Instead of using the reduced widths defined in Eq. (25), it is possible to express the conductance in terms of the widths for the maximally-projected spin states (i.e. $M = S$ and $M' = S'$)

$$\Gamma_{\alpha S,\alpha'S'}^{\mathrm{l,r}} \equiv \Gamma_{\alpha SS,\alpha'S'S'}^{\mathrm{l,r}} . \qquad (29)$$

The reduced matrix elements in Eq. (25) are non-zero only for $S' = S \pm 1/2$ and, using the corresponding CG coefficients, we have

$$\tilde{\Gamma}_{\alpha S,\alpha'S'}^{\mathrm{l,r}} = [2\max(S,S') + 1]\,\Gamma_{\alpha S,\alpha'S'}^{\mathrm{l,r}} . \qquad (30)$$

VI. EXPLICIT SOLUTIONS IN THE LIMIT OF ELASTIC SCATTERING

In general, it is not possible to find a closed solution for the conductance in the limit of elastic scattering, and Eqs. (16) have to be solved numerically. However, in the following we list four important cases where explicit solutions to Eqs. (24) or more generally to Eqs. (16) exist and the conductance is given by a closed expression.

A. Ground-state transition at low temperatures

At low enough temperatures, only the transition between the ground states of the n- and $(N + 1)$-electron dots $[(N,S) \rightarrow (N+1,S')]$ provides an important contribution to the conductance. It is then sufficient to consider the two ground-state manifolds (N,S) and $(N+1,S')$ which are respectively $(2S+1)$- and $(2S'+1)$-fold degenerate. In this case, there is only one equation (24) and a solution can always be found;[14] the two variables $\Psi_S^{(N)}$ and $\Psi_{S'}^{(N+1)}$ are uniquely determined by

$$\Psi_{S'}^{(N+1)} = \Psi_S^{(N)} - \frac{\eta_{\mathrm{l}}\tilde{\Gamma}_{S,S'}^{\mathrm{l}} - \eta_{\mathrm{r}}\tilde{\Gamma}_{S,S'}^{\mathrm{r}}}{\tilde{\Gamma}_{S,S'}^{\mathrm{l}} + \tilde{\Gamma}_{S,S'}^{\mathrm{r}}} \qquad (31)$$

and by the normalization condition (17). The linear conductance then yields

$$G_{\mathrm{el}} = \frac{e^2}{\hbar kT} \frac{[2\max(S,S')+1]f(\varepsilon_{S'}^{(N+1)} - \varepsilon_S^{(N)} - \tilde{\epsilon}_{\mathrm{F}})}{(2S+1) + (2S'+1)e^{-\beta(\varepsilon_{S'}^{(N+1)} - \varepsilon_S^{(N)} - \tilde{\epsilon}_{\mathrm{F}})}} \frac{\Gamma_{S,S'}^{\mathrm{l}}\Gamma_{S,S'}^{\mathrm{r}}}{\Gamma_{S,S'}^{\mathrm{l}} + \Gamma_{S,S'}^{\mathrm{r}}} , \qquad (32)$$

where the widths $\Gamma_{S,S'}^{\mathrm{l,r}}$ correspond to the maximally-projected spin states with $M = S$ and $M' = S'$

[cf. Eqs. (29) and (30)]. According to Eq. (32), the func-

tional form of the conductance (versus the effective Fermi energy) does not depend on tunneling widths. In particular, G_{el} is maximized when the effective Fermi energy $\tilde{\epsilon}_{\text{F}} \equiv e\zeta V_{\text{g}} + \epsilon_{\text{F}}$ is tuned to

$$\tilde{\epsilon}_{\text{F,max}} = \varepsilon_{S'}^{(N+1)} - \varepsilon_S^{(N)} - \frac{kT}{2}\ln\left(\frac{2S'+1}{2S+1}\right) . \quad (33)$$

B. Ground-state–to–excited-states transitions

We consider only a single state in either the N- or the $(N+1)$-electron system and allow for any number of states in the other system. This is useful at low temperatures, when for an even number of electrons the lowest states in the spin sectors of $S = 0$ and $S = 1$ can be close in energy. The approximation of the previous paragraph may then be poor even at very low temperatures. For example, one might consider the transitions $(N, S = 1/2) \to (N+1, \alpha', S')$ with $S' = 0$ and 1. The variables associated to the $(N+1)$-electrons state are chosen as

$$\Psi_{\alpha'S'}^{(N+1)} = \Psi_S^{(N)} - \frac{\eta_{\text{l}}\tilde{\Gamma}_{S,\alpha'S'}^{\text{l}} - \eta_{\text{r}}\tilde{\Gamma}_{S,\alpha'S'}^{\text{r}}}{\tilde{\Gamma}_{S,\alpha'S'}^{\text{l}} + \tilde{\Gamma}_{S,\alpha'S'}^{\text{r}}} , \quad (34)$$

and the variable $\Psi_S^{(N)}$ is then fixed by the normalization condition (17). Clearly, Eqs. (34) solve Eqs. (24) for all α' and the conductance has the form (28).

C. Systems with good orbital occupation numbers

We define $n_\lambda = n_{\lambda+} + n_{\lambda-}$ to be the total occupation of the (doubly-degenerate) single-particle orbital λ, i.e. the sum of the number of spin-up and spin-down particles in that orbital. Both $n_{\lambda+}$ and $n_{\lambda-}$ can take the value 0 or 1, hence n_λ obtains the values 0, 1 or 2. We will show below that Eqs. (26) have a solution when all n_λ are good quantum numbers. For a given pair (α, S) and (α', S') only one term $\lambda = \lambda_0$ in $\psi_\sigma^\dagger(\mathbf{r}) = \sum_\lambda \psi_\lambda(\mathbf{r})a_{\lambda\sigma}^\dagger$ will then contribute to the respective matrix element

$$(\alpha'S'\|\psi^\dagger(\mathbf{r})\|\alpha S) = \sum_\lambda \psi_\lambda(\mathbf{r})(\alpha'S'\|a_\lambda^\dagger\|\alpha S)$$
$$= \psi_{\lambda_0}(\mathbf{r})(\alpha'S'\|a_{\lambda_0}^\dagger\|\alpha S). \quad (35)$$

Clearly, if more than one orbital λ contribute to the sum in Eq. (35), then the occupations cannot be good quantum numbers in the final state α'. Since the dependence of the reduced matrix element in Eq. (35) on the point contact at $\mathbf{r}$ is only through the wave function $\psi_{\lambda_0}(\mathbf{r})$, we have

$$\frac{\eta_{\text{l}}\tilde{\Gamma}_{\alpha S,\alpha'S'}^{\text{l}} - \eta_{\text{r}}\tilde{\Gamma}_{\alpha S,\alpha'S'}^{\text{r}}}{\tilde{\Gamma}_{\alpha S,\alpha'S'}^{\text{l}} + \tilde{\Gamma}_{\alpha S,\alpha'S'}^{\text{r}}} = \frac{\eta_{\text{l}}\Gamma_{\lambda_0}^{\text{l}} - \eta_{\text{r}}\Gamma_{\lambda_0}^{\text{r}}}{\Gamma_{\lambda_0}^{\text{l}} + \Gamma_{\lambda_0}^{\text{r}}} , \quad (36)$$

where

$$\Gamma_\lambda^{\text{l,r}} = \Gamma_0^{\text{l,r}}|\psi_\lambda(\mathbf{r}_{\text{l,r}})|^2 \quad (37)$$

are the single-particle decay widths.

We now argue that Eqs. (24) have a solution given by

$$\Psi_{\alpha S}^{(m)} = -\sum_\lambda \frac{\eta_{\text{l}}\Gamma_\lambda^{\text{l}} - \eta_{\text{r}}\Gamma_\lambda^{\text{r}}}{\Gamma_\lambda^{\text{l}} + \Gamma_\lambda^{\text{r}}}n_\lambda(\alpha, S)$$
$$\text{for } m = N, N+1 , \quad (38)$$

where $n_\lambda(\alpha, S)$ are the single-particle level occupation numbers of the states with quantum numbers α and S [note that the label α includes all the good quantum numbers $\{n_\lambda\} = (n_1, n_2, \ldots)$ plus any additional quantum numbers required to distinguish between states with the same occupations $\{n_\lambda\}$ and spin S]. Since two manifolds of states (α, S) and (α', S') can only be connected by changing the occupation of a particular orbital λ_0 by a single particle, we have

$$\Psi_{\alpha S}^{(N)} - \Psi_{\alpha'S'}^{(N+1)}$$
$$= \sum_\lambda \frac{\eta_{\text{l}}\Gamma_\lambda^{\text{l}} - \eta_{\text{r}}\Gamma_\lambda^{\text{r}}}{\Gamma_\lambda^{\text{l}} + \Gamma_\lambda^{\text{r}}}[n_\lambda(\alpha, S) - n_\lambda(\alpha', S')]$$
$$= \frac{\eta_{\text{l}}\Gamma_{\lambda_0}^{\text{l}} - \eta_{\text{r}}\Gamma_{\lambda_0}^{\text{r}}}{\Gamma_{\lambda_0}^{\text{l}} + \Gamma_{\lambda_0}^{\text{r}}} . \quad (39)$$

Thus Eqs. (26) can be indeed satisfied by (38). For the conductance, we obtain the expression

$$G = \frac{e^2}{\hbar kT}\sum_{\substack{\alpha S \\ \alpha'S'}} \tilde{P}_{\alpha S}^{(N)}f(\varepsilon_{\alpha S,\alpha'S'})$$
$$\times |(\alpha'S'\|a_{\lambda_0}^\dagger\|\alpha S)|^2\frac{\Gamma_{\lambda_0}^{\text{l}}\Gamma_{\lambda_0}^{\text{r}}}{\Gamma_{\lambda_0}^{\text{l}} + \Gamma_{\lambda_0}^{\text{r}}} , \quad (40)$$

where the orbital λ_0 depends on both the manifolds (α, S) and (α', S'). Of course, only manifolds that are connected by the addition of an electron to a single-particle orbital contribute to the sum in Eq. (40); i.e. the occupations $\{n_\lambda(\alpha', S')\}$ of the manifold (α', S') can only differ for one orbital and by one unit from the occupations $\{n_\lambda(\alpha, S)\}$ of the manifold (α, S). An expression similar to (40) (but including explicit sums over the magnetic quantum numbers) was discussed in Ref. 13.

An important example of a Hamiltonian in the class discussed here is the so-called universal Hamiltonian in the limit of the Thouless conductance $g_{\text{T}} \to \infty$,

$$\hat{H} = \sum_{\lambda\sigma} \epsilon_\lambda a_{\lambda\sigma}^\dagger a_{\lambda\sigma} + \frac{e^2}{2C}\hat{n}^2 - J_S\hat{\mathbf{S}}^2 , \quad (41)$$

where $\hat{\mathbf{S}} = \sum_\lambda \hat{\mathbf{S}}_\lambda$ with $\hat{\mathbf{S}}_\lambda = \sum_{\sigma\sigma'} a_{\lambda\sigma}^\dagger \boldsymbol{\sigma}_{\sigma\sigma'} a_{\lambda\sigma'}$ is the total-spin operator of the dot and $\boldsymbol{\sigma}$ is the vector of the three 2×2 Pauli matrices. Since $[\hat{n}_\lambda, \hat{\mathbf{S}}_\lambda] = 0$, the Hamiltonian (41) is invariant under spin rotations and characterized by the good quantum numbers $\{n_\lambda\}, S$, and M. The conductance through a weakly-coupled dot with a Hamiltonian (41) is then given by Eq. (40).

D. Systems with degenerate ground states

In this subsection, we consider the case of low temperatures, when only transitions between the ground states with N and $N + 1$ electrons in the dot are allowed, and these ground states are M_N- and M_{N+1}-fold degenerate, respectively. This case is more general than the similar case of Section VI A, since the degeneracy is not necessarily the usual spin degeneracy, and the transition widths $\Gamma_{ij}^{\mathrm{l,r}}$ can be arbitrary and are not necessarily related by a Wigner-Eckart theorem. Although we cannot write a closed solution for G_{el} in this more general case, the degeneracy of the levels participating in the transport process allows for significant simplification of the expression for G_{el} [as compared with the general expression given by Eqs. (16), (17), and (20)].

The level degeneracy allows us to reduce the rate equations (16) to

$$\sum_j [(\Gamma_{ij}^{\mathrm{l}} + \Gamma_{ij}^{\mathrm{r}})(\Psi_j^{(N+1)} - \Psi_i^{(N)})$$

$$+ (\eta_{\mathrm{l}} \Gamma_{ij}^{\mathrm{l}} - \eta_{\mathrm{r}} \Gamma_{ij}^{\mathrm{r}})] = 0 \quad \text{for all } i, \text{ (42a)}$$

$$\sum_i [(\Gamma_{ij}^{\mathrm{l}} + \Gamma_{ij}^{\mathrm{r}})(\Psi_j^{(N+1)} - \Psi_i^{(N)})$$

$$+ (\eta_{\mathrm{l}} \Gamma_{ij}^{\mathrm{l}} - \eta_{\mathrm{r}} \Gamma_{ij}^{\mathrm{r}})] = 0 \quad \text{for all } j. \text{ (42b)}$$

As mentioned in Section IV, only $M_N + M_{N+1} - 1$ of these $M_N + M_{N+1}$ equations are linearly independent. The equation needed to determine a unique solution is given by Eq. (17). In the special case discussed here, the equilibrium probabilities for N and $N + 1$ electrons are independent of the states i and j, respectively, and Eq. (17) simplifies to

$$\tilde{P}^{(N)} \sum_i \Psi_i^{(N)} + \tilde{P}^{(N+1)} \sum_j \Psi_j^{(N+1)} = 0 , \qquad (43)$$

where we have defined $\tilde{P}^{(N)} = \tilde{P}_i^{(N)}$ for all i and $\tilde{P}^{(N+1)} = \tilde{P}_j^{(N+1)}$ for all j. In particular

$$\tilde{P}^{(N)} = \frac{1}{M_N + M_{N+1} e^{-\beta(\varepsilon^{(N+1)} - \varepsilon^{(N)} - \tilde{\epsilon}_{\mathrm{F}})}} , \qquad (44)$$

where $\varepsilon^{(N)} = \varepsilon_i^{(N)}$ for all i and $\varepsilon^{(N+1)} = \varepsilon_j^{(N+1)}$ for all j.

Using Eq. (44), the general expression (20) for the conductance reduces to

$$G_{\mathrm{el}} = \frac{e^2}{\hbar kT} \frac{f(\varepsilon^{(N+1)} - \varepsilon^{(N)} - \tilde{\epsilon}_{\mathrm{F}})}{M_N + M_{N+1} e^{-\beta(\varepsilon^{(N+1)} - \varepsilon^{(N)} - \tilde{\epsilon}_{\mathrm{F}})}}$$

$$\times \sum_{ij} (\Psi_i^{(N)} - \Psi_j^{(N+1)} - \eta_{\mathrm{l}}) \Gamma_{ij}^{\mathrm{l}} . \qquad (45)$$

Here, $\Psi_i^{(N)}$ and $\Psi_j^{(N+1)}$ are a solution to Eqs. (42). These equations do not depend on the temperature, and we can therefore choose a private solution $\tilde{\Psi}_i^{(N)}$ and $\tilde{\Psi}_j^{(N+1)}$ that is temperature-independent and determined solely by the tunneling widths. The general solution of Eqs. (42) is given by $\Psi_i^{(N)} = \tilde{\Psi}_i^{(N)} + c$ and

$\Psi_j^{(N+1)} = \tilde{\Psi}_j^{(N+1)} + c$, where c is a constant. The solution that satisfies the normalization condition (43) is the one with $c = -(\tilde{P}^{(N)} \sum_i \tilde{\Psi}_i^{(N)} + \tilde{P}^{(N+1)} \sum_j \tilde{\Psi}_j^{(N+1)})$. While this constant c depends on temperature, it drops out in the final expression for the conductance, thus making the sum on the r.h.s. of Eq. (45) temperature-independent.

The conductance in Eq. (45) factorizes into two contributions. The sum over i and j is completely determined by the tunneling widths $\Gamma_{ij}^{\mathrm{l,r}}$ and is independent of temperature and the effective Fermi energy (it is also independent of η_{l} and η_{r} as is shown in the Appendix). The prefactor of the sum on the r.h.s. of Eq. (45) does not depend on the tunneling widths $\Gamma_{ij}^{\mathrm{l,r}}$ and contains the full dependence on the temperature and Fermi energy. This prefactor determines the functional dependence of the conductance peak on the gate voltage. Similar to Eq. (33) in Section VI A, the maximum of G_{el} is attained when the effective Fermi energy is tuned to

$$\tilde{\epsilon}_{\mathrm{F,max}} = \varepsilon^{(N+1)} - \varepsilon^{(N)} - \frac{kT}{2} \ln \left(\frac{M_{N+1}}{M_N} \right) . \qquad (46)$$

Eqs. (42) need to be solved numerically. An exception is the case discussed in Section VI A, where the dot has spin-rotation symmetry and the degeneracy of the ground state corresponds to the various values of the spin projection M. In this case Eqs. (42) can be solved in closed form [see Eq. (31)], and the linear conductance (45) reduces then to Eq. (32).

VII. THE RAPID-THERMALIZATION LIMIT

The above derivation of the linear conductance has been done under the assumption of dominantly elastic scattering, which is a good approximation at sufficiently low temperatures. Although the exact temperature dependence of inelastic scattering events is not well understood, they should become more relevant with increasing temperature, such that their width Γ_{in} will eventually be of comparable size to Γ_{el}. We therefore consider in this Section the rapid-thermalization limit $\Gamma_{\mathrm{in}} \gg \Gamma_{\mathrm{el}}$.

We denote by $P(N)$ the probability of the dot to be in a N-electron state, and by $P(i|N)$ the conditional probability of the dot to be in a particular many-body state i given the dot is occupied by N electrons. In general

$$P_i^{(N)} = P(i|N)P(N) . \qquad (47)$$

In the rapid-thermalization limit, the conditional probabilities $P(i|N)$ and $P(j|N + 1)$ are always given by their thermal equilibrium values $\tilde{P}(i|N)$ and $\tilde{P}(j|N + 1)$, respectively. It is then only the probabilities $P(N)$ and $P(N + 1)$ that obtain non-equilibrium values by a finite bias voltage V. These probabilities satisfy the following rate equations:

$$\frac{\partial P(N)}{\partial t} = P(N+1)\sum_{i,j}\tilde{P}(j|N+1)[(1-f_{ij}^{\mathrm{l}})\Gamma_{ij}^{\mathrm{l}} + (1-f_{ij}^{\mathrm{r}})\Gamma_{ij}^{\mathrm{r}}] - P(N)\sum_{i,j}\tilde{P}(i|N)[f_{ij}^{\mathrm{l}}\Gamma_{ij}^{\mathrm{l}} + f_{ij}^{\mathrm{r}}\Gamma_{ij}^{\mathrm{r}}]\,, \quad (48\text{a})$$

$$\frac{\partial P(N+1)}{\partial t} = P(N)\sum_{i,j}\tilde{P}(i|N)[f_{ij}^{\mathrm{l}}\Gamma_{ij}^{\mathrm{l}} + f_{ij}^{\mathrm{r}}\Gamma_{ij}^{\mathrm{r}}] - P(N+1)\sum_{i,j}\tilde{P}(j|N+1)[(1-f_{ij}^{\mathrm{l}})\Gamma_{ij}^{\mathrm{l}} + (1-f_{ij}^{\mathrm{r}})\Gamma_{ij}^{\mathrm{r}}]\,. \quad (48\text{b})$$

As in the elastic case, we are interested in stationary solutions $\partial P(N)/\partial t = \partial P(N+1)/\partial t = 0$, where the rate at which electrons tunnel onto the dot $(N \rightarrow N+1)$ is equal to the rate at which electrons tunnel off the dot $(N+1 \rightarrow N)$. For a small voltage V, we can expand these probabilities around their equilibrium values $\tilde{P}(N)$ and $\tilde{P}(N+1)$ to first order in V,

$$P(N) = \tilde{P}(N)[1 + eV\beta\Phi^{(N)}]\,, \quad (49\text{a})$$

$$P(N+1) = \tilde{P}(N+1)[1 + eV\beta\Phi^{(N+1)}]\,, \quad (49\text{b})$$

where $\Phi^{(N)}$ and $\Phi^{(N+1)}$ are unknown variables. We ob-serve that while in the elastic limit there were $M_N + M_{N+1}$ unknown variables, in the inelastic limit we are left with a considerably simpler situation of only two un-known variables. Consequently, an explicit expression for the conductance can always be given in the inelastic limit.

In analogy to the elastic case, we proceed by substitut-ing linear expansions in Eqs. (8) and (49) into Eqs. (48), and imposing stationarity. Collecting the terms that are independent of the bias voltage V, we obtain the usual detailed-balance equation at equilibrium. Keeping only terms linear in V, we arrive at

$$\tilde{P}(N+1)\sum_{ij}\tilde{P}(j|N+1)[\beta(1-f_{ij})(\Gamma_{ij}^{\mathrm{l}} + \Gamma_{ij}^{\mathrm{r}})\Phi^{(N+1)} - f_{ij}'(\eta_{\mathrm{l}}\Gamma_{ij}^{\mathrm{l}} - \eta_{\mathrm{r}}\Gamma_{ij}^{\mathrm{r}})]$$

$$-\tilde{P}(N)\sum_{ij}\tilde{P}(i|N)[\beta f_{ij}(\Gamma_{ij}^{\mathrm{l}} + \Gamma_{ij}^{\mathrm{r}})\Phi^{(N)} + f_{ij}'(\eta_{\mathrm{l}}\Gamma_{ij}^{\mathrm{l}} - \eta_{\mathrm{r}}\Gamma_{ij}^{\mathrm{r}})] = 0\,. \quad (50)$$

Using the equilibrium relations $\tilde{P}(N)\tilde{P}(i|N) = \tilde{P}_i^{(N)}$ and $\tilde{P}(N+1)\tilde{P}(j|N+1) = \tilde{P}_j^{(N+1)}$, together with Eqs. (13), (14), and (15), we can rewrite Eq. (50) in the form

$$\sum_{ij}\tilde{P}_i^{(N)}f_{ij}\Big[(\Phi^{(N+1)} - \Phi^{(N)})(\Gamma_{ij}^{\mathrm{l}} + \Gamma_{ij}^{\mathrm{r}})$$

$$-(\eta_{\mathrm{l}}\Gamma_{ij}^{\mathrm{l}} - \eta_{\mathrm{r}}\Gamma_{ij}^{\mathrm{r}})\Big] = 0\,. \quad (51)$$

The solution for $\Phi^{(N)}$ and $\Phi^{(N+1)}$ (up to an arbitrary additive constant) is then given by

$$\Phi^{(N+1)} - \Phi^{(N)} = \frac{\sum_{ij}P_i^{(N)}f_{ij}(\eta_{\mathrm{l}}\Gamma_{ij}^{\mathrm{l}} - \eta_{\mathrm{r}}\Gamma_{ij}^{\mathrm{r}})}{\sum_{ij}P_i^{(N)}f_{ij}(\Gamma_{ij}^{\mathrm{l}} + \Gamma_{ij}^{\mathrm{r}})}\,. \quad (52)$$

The additive constant is determined by the normalization condition $\tilde{P}(N)\Phi^{(N)} + \tilde{P}(N+1)\Phi^{(N+1)} = 0$.

To find the linear conductance G_{therm} in the limit of rapid-thermalization, we substitute the expansions in Eqs. (8) and (49) in the general expression for the linear-response current in Eq. (19). We obtain

$$G_{\mathrm{therm}} = \frac{e^2}{\hbar kT}\sum_{ij}\tilde{P}_i^{(N)}f_{ij}(\Phi^{(N)} - \Phi^{(N+1)} - \eta_{\mathrm{l}})\Gamma_{ij}^{\mathrm{l}}\,. \quad (53)$$

With the solution in Eq. (52), we obtain a closed expres-sion for the rapid-thermalization conductance

$$G_{\mathrm{therm}} = \frac{e^2}{\hbar kT}\frac{\left(\sum_{ij}\tilde{P}_i^{(N)}f_{ij}\Gamma_{ij}^{\mathrm{l}}\right)\left(\sum_{kl}\tilde{P}_k^{(N)}f_{kl}\Gamma_{ij}^{\mathrm{r}}\right)}{\sum_{rs}\tilde{P}_r^{(N)}f_{rs}(\Gamma_{rs}^{\mathrm{l}} + \Gamma_{rs}^{\mathrm{r}})}\,. \quad (54)$$

As in the elastic case, we can exploit the spin symme-tries to perform explicitly the summation over the spin projections M and M',

$$\sum_{ij}\tilde{P}_i^{(N)}f_{ij}\Gamma_{ij}^{\mathrm{l,r}} = \sum_{\alpha S\alpha'S'}\tilde{P}_{\alpha S}^{(N)}f(\varepsilon_{\alpha'S',\alpha S})\tilde{\Gamma}_{\alpha S,\alpha'S'}^{\mathrm{l,r}}\,. \quad (55)$$

This yields as a final result for the conductance in the case of strong inelastic scattering

$$G_{\text{therm}} = \frac{e^2}{\hbar kT} \frac{\left(\sum\limits_{\alpha S \alpha' S'} \tilde{P}_{\alpha S}^{(N)} f(\varepsilon_{\alpha S, \alpha' S'}) \tilde{\Gamma}_{\alpha S \alpha' S'}^{\text{l}}\right) \left(\sum\limits_{\alpha S \alpha' S'} \tilde{P}_{\alpha S}^{(N)} f(\varepsilon_{\alpha S, \alpha' S'}) \tilde{\Gamma}_{\alpha S \alpha' S'}^{\text{r}}\right)}{\left(\sum\limits_{\alpha S \alpha' S'} \tilde{P}_{\alpha S}^{(N)} f(\varepsilon_{\alpha S, \alpha' S'}) (\tilde{\Gamma}_{\alpha S \alpha' S'}^{\text{l}} + \tilde{\Gamma}_{\alpha S \alpha' S'}^{\text{r}})\right)} . \tag{56}$$

VIII. SUMMARY AND CONCLUSION

We define

$$\langle\!\langle X_{ij} \rangle\!\rangle \equiv \sum_{ij} \tilde{P}_i^{(N)} f(\varepsilon_{ij}) X_{ij}$$
$$= \sum_{ij} \tilde{P}_j^{(N+1)} [1 - f(\varepsilon_{ij})] X_{ij} , \tag{57}$$

where X_{ij} is a quantity that depends on the many-body states i and j of the N- and $(N+1)$-electron dots, respectively. The equilibrium probabilities $\tilde{P}_i^{(N)}$ and $\tilde{P}_j^{(N+1)}$ to find the dot in states i and j are given by Eqs. (10) and (11), while the energy difference ε_{ij} is defined in Eq. (5).

In the rapid-thermalization limit, the conductance of the almost-isolated dot is given by

$$G_{\text{therm}} = \frac{e^2}{\hbar kT} \frac{\langle\!\langle \Gamma_{ij}^{\text{l}} \rangle\!\rangle \langle\!\langle \Gamma_{ij}^{\text{r}} \rangle\!\rangle}{\langle\!\langle \Gamma_{ij}^{\text{l}} + \Gamma_{ij}^{\text{r}} \rangle\!\rangle} , \tag{58}$$

where Γ_{ij}^{l} (Γ_{ij}^{r}) are the partial transition widths between the states i and j involving the tunneling of an electron from the left (right) lead into the dot.

In the elastic limit, a closed solution is not possible in the general case. The conductance can be calculated by solving the linear equations (16) for $\Psi_i^{(N)}$ and $\Psi_j^{(N+1)}$ and then substituting in the expression (20). However, there are several important cases where a closed solution is possible, including the constant-exchange-interaction model (i.e. the universal Hamiltonian in the limit of infinite Thouless conductance). In these cases

$$G_{\text{el}} = \frac{e^2}{\hbar kT} \left\langle\!\!\left\langle \frac{\Gamma_{ij}^{\text{l}} \Gamma_{ij}^{\text{r}}}{\Gamma_{ij}^{\text{l}} + \Gamma_{ij}^{\text{r}}} \right\rangle\!\!\right\rangle . \tag{59}$$

When the Hamiltonian is invariant under spin rotations, these expressions can be simplified by carrying out explicitly the summation over the magnetic quantum numbers. The many-body levels of the N and $(N + 1)$-electron dots are now characterized by the quantum numbers α, S, M and α', S', M' and the respective energies are independent of M and M'. Equations (58) and (59) are now valid with the reduced widths $\tilde{\Gamma}_{\alpha S \alpha' S'}^{\text{l,r}}$ replacing the widths $\tilde{\Gamma}_{\alpha S M \alpha' S' M'}^{\text{l,r}}$, and the summation in Eq. (57) carried over αS and $\alpha' S'$ only (but not over M and M').

In conclusion, we have solved the rate equations and found the linear conductance in the presence of interactions in the dot (beyond the charging energy). In particular, we have taken into account the spin degrees of freedom of the dot and we showed the simplifications that occur when the dot's Hamiltonian is invariant under spin rotations. Both the limits of dominantly elastic scattering and rapid thermalization are discussed. This work generalizes the solutions of Ref. 4 that were derived in the limit of non-interacting electrons (except for a constant charging energy).

Appendix

Here we show that the linear conductance in Eq. (20) is independent of η_{l} and η_{r}, the fractions of the bias potential difference between the dot and the corresponding leads. Defining the new variables $\Phi_i^{(N)} \equiv \Psi_i^{(N)} - \eta_{\text{l}}$, we can rewrite Eqs. (16) in the form

$$\sum_j f_{ij}[(\Gamma_{ij}^{\text{l}} + \Gamma_{ij}^{\text{r}})(\Psi_j^{(N+1)} - \Phi_i^{(N)}) - \Gamma_{ij}^{\text{r}}] = 0$$
$$\text{for all } i , \tag{60a}$$

$$\sum_i (1 - f_{ij})[(\Gamma_{ij}^{\text{l}} + \Gamma_{ij}^{\text{r}})(\Psi_j^{(N+1)} - \Phi_i^{(N)}) - \Gamma_{ij}^{\text{r}}] = 0$$
$$\text{for all } j . \tag{60b}$$

The solution of Eqs. (60) is determined up to an additive constant. Since η_{l} and η_{r} do not appear explicitly in Eqs. (60), we can choose a private solution $\tilde{\Phi}_i^{(N)}$ and $\tilde{\Psi}_j^{(N+1)}$ that is independent of η_{l} and η_{r}. The general solution is then given by $\Phi_i^{(N)} = \tilde{\Phi}_i^{(N)} + c$ and $\Psi_j^{(N+1)} = \tilde{\Psi}_j^{(N+1)} + c$, where c is a constant. In particular, the normalization condition (17) can be satisfied by choosing

$$c = -\left[\sum_i \tilde{P}_i^{(N)} (\tilde{\Phi}_i^{(N)} + \eta_{\text{l}}) + \sum_j \tilde{P}_j^{(N+1)} \tilde{\Psi}_j^{(N+1)}\right] . \tag{61}$$

The constant c depends on η_l, but disappears in the final expression for the conductance in Eq. (20),

$$G_{el} = \frac{e^2}{\hbar kT} \sum_{ij} \tilde{P}_i^{(N)} f_{ij} (\tilde{\Phi}_i^{(N)} - \tilde{\Psi}_j^{(N+1)}) \Gamma_{ij}^l. \qquad (62)$$

Expression (62) for the conductance shows clearly its independence of η_l (and η_r).

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